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On the propagation of small disturbances in a relaxing gas with heat addition

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General solutions for the linearized three-dimensional unsteady motion of a relaxing gas, including the effects of direct heat addition, are obtained using a Green's function technique. The possibility of direct heat addition into each separate mode of energy storage must be catered for, and the solutions have some bearing on the question of the heating of a gas by radiation.

Examples are given of initial-value and heat-source solutions in an unbounded domain and the boundary-value problem is exemplified by a study of the motion created by a spherical piston.

1. Introduction

Solutions for the linearized flow of a relaxing gas have previously been obtained for a variety of different situations, but not so far for the general case of three-dimensional unsteady motions. The present paper is therefore directly concerned with this case and, using the Green's function approach, derives results of considerable generality.

The opportunity has also been taken to include the possibility of direct heat addition to the gas, assuming that the heat addition terms are known functions of space and time. For simplicity we shall only deal with one relaxation process so that the pure gas is assumed to carry communicable energy in the translational mode of molecular motion, and in one (relaxing) internal mode. The existence of these two modes of energy storage makes it necessary for us to distinguish between the possibilities of direct heat addition into each mode separately. Aside from the inhomogeneous term (which will be found to arise from heat addition), the basic non-equilibrium equation for the potential function has been established previously in quite general terms (e.g. Vincenti 1959). At the risk of being a little repetitious therefore, we have re-derived the equation in some detail in order to bring out as clearly as possible the role of (and the assumptions made about) the heat addition terms.

One may question the practicality of the direct heat addition idea, and indeed we have something to say about this matter in relation to the translational states in § 6. With regard to the internal mode, however, we may readily excuse the notion by appealing to the physical ideas behind the radiant heating processes which take place in a gas. Briefly, we can take it that the capture of a photon by a gas molecule will lead, in the first instance, to a change in that molecule's internal state. The subsequent (collisional) relaxation processes will then distribute this internal mode energy throughout the gas. It goes without saying that the true state of affairs in a radiating gas does not generally admit the possibility of our knowing the form of the heat addition terms at the outset; they are indeed most intimately connected with the behaviour of the gas itself. We can perhaps excuse the present idealization by viewing it as a small step in the direction of a more complete and detailed treatment of the non-equilibrium radiation problem in gas-dynamics.

Some special cases of the general solution are considered at the end of the paper. The unbounded-domain Green's function is derived (using integral transform methods) and in consequence the initial-value and heat-source problems in a gas devoid of solid boundaries can be taken as solved. To illustrate the boundary-value problem, the simple case of a spherically expanding piston is considered. The Green's function necessary for this solution is found directly by integral transform methods.

2. The basic equations

The five basic conservation equations, namely those of mass, momentum and energy are, in order,

$$\frac{D\rho}{Dt}+\rho\nabla.\mathbf{u} = 0, \tag{1}$$

$$\frac{D\mathbf{u}}{Dt}+\frac{1}{\rho}\nabla p = 0, \tag{2}$$

$$\frac{De}{Dt}+\frac{p}{\rho}\nabla.\mathbf{u} = q. \tag{3}$$

where D/Dt is the usual convective operator and ∇ is the (vector) gradient operator. p and ρ are pressure and density respectively, $\mathbf{u}$ is the velocity vector and e the specific internal energy; q is the *total* quantity of heat added per unit mass of gas per unit time. On the assumption that the gas molecules have but one internal energy mode, we shall write the caloric equation of state in the form

$$e = C_{v1}T_1+C_2T_2. \tag{4}$$

The thermal equation of state is

$$p = \rho RT_1. \tag{5}$$

T_1 and T_2 are the translational- and internal-mode temperatures respectively; they are not equal in general. C_2 is the specific heat of the internal energy mode whilst C_{v1} is the specific heat at constant volume of the translational mode of motion. We assume that both quantities are constants. R is the gas constant for the particular (pure) gas in question.

Equations (1) to (5) inclusive represent seven equations for the eight unknown quantities p, ρ, T_1, T_2, e and $\mathbf{u}$, so that a further equation is required. This is the relaxation equation and, because of the slight degree of novelty arising from the possibility that heat may be added directly to the internal mode alone, we shall go into the details of its derivation here. One method is as follows.

Let n_i be the number density of gas molecules in an internal quantum state i. The continuity equation for molecules of this type can be written in the form

$$Dn_i/Dt + n_i \nabla . \mathbf{u} = \dot{N}_i \tag{6}$$

(see Clarke & McChesney 1964, equation (5.3.7)). $\dot{N}_i$ is the number-rate of production of molecules in state i per unit volume per unit time and diffusion effects have been neglected. If ϵ_i is the (constant) energy in the internal quantum state i we can multiply (6) throughout by ϵ_i and sum over all the allowable states. Then we can write

$$\sum_i \epsilon_i n_i = \rho e_2 = \rho C_2 T_2, \tag{7}$$

where e_2 ($= C_2 T_2$) is the total energy contained in the internal mode in unit mass of gas (see (4), *et seq.*). Using expression (7) in (6) and making use of (1) it is easy to verify that

$$\frac{De_2}{Dt} = \frac{1}{\rho} \sum_i \epsilon_i \dot{N}_i. \tag{8}$$

The right-hand side of (8) represents the total rate of gain of energy by the internal mode per unit mass. We divide up this latter quantity into two parts. The first arises from excitation of the internal mode by molecular collisions, with a relaxation time τ' (assumed constant). This part may be adequately represented by the expression

$$\{e_2(T_1) - e_2(T_2)\}/\tau' = C_2\{T_1 - T_2\}/\tau'.$$

The second part is just the heat added to the mode per unit mass per unit time, which we write here as q_2, assuming it to be a known function in the same way as q in (3) above.

It follows that the desired relaxation equation is

$$\tau' DT_2/Dt + T_2 - T_1 = \tau' q_2/C_2. \tag{9}$$

Now if C_{p1} is the specific heat of the translational mode at constant pressure, so that

$$C_{p1} - C_{v1} = R, \quad C_{p1}/C_{v1} = \gamma_1, \tag{10}$$

we can rewrite (4), with the aid of (5), in the form

$$e = \frac{1}{\gamma_1 - 1} \frac{p}{\rho} + C_2 T_2. \tag{11}$$

Using (1) it follows that (3) can be re-expressed in the form

$$\frac{1}{\rho}\frac{Dp}{Dt} + a_f^2 \nabla . \mathbf{u} + C_2(\gamma_1 - 1)\frac{DT_2}{Dt} = (\gamma_1 - 1) q, \tag{12}$$

where we have written

$$a_f^2 = \gamma_1 p/\rho, \tag{13}$$

and a_f is the frozen speed of sound.

Without going into the details, one can now use (3), (4) and (9) to eliminate the translational temperature T_1 and, subsequently, use (12) to eliminate the remaining derivatives of T_2. The result is the following equation:

$$\tau' \frac{D}{Dt}\left\{\frac{1}{\rho}\frac{Dp}{Dt} + a_f^2 \nabla . \mathbf{u}\right\} + \frac{C_v}{C_{v1}}\left\{\frac{1}{\rho}\frac{Dp}{Dt} + a_e^2 \nabla . \mathbf{u}\right\} = (\gamma_1 - 1)\left\{\tau' \frac{D}{Dt}(q - q_2) + q\right\}, \tag{14}$$

Equation (23) for the dimensionless disturbance potential therefore has the form

$$L[\phi] = -4\pi Q. \tag{27}$$

The Green's function $G(\mathbf{r}, t \mid \mathbf{r}_0, t_0)$, representing the effect at a field-point $\mathbf{r}$, t arising from a source located at a source-point $\mathbf{r}_0$, t_0, satisfies the equation

$$L[G(\mathbf{r}, t \mid \mathbf{r}_0, t_0)] = -4\pi\delta(\mathbf{r}-\mathbf{r}_0)\,\delta(t-t_0), \tag{28}$$

whilst the adjoint Green's function $\tilde{G}(\mathbf{r}, t \mid \mathbf{r}_0, t_0)$ is found by solving the adjoint equation

$$\tilde{L}[\tilde{G}(\mathbf{r}, t \mid \mathbf{r}_0, t_0)] = -4\pi\delta(\mathbf{r}-\mathbf{r}_0)\,\delta(t-t_0). \tag{29}$$

δ is the Dirac function and, in particular, $\delta(\mathbf{r}-\mathbf{r}_0)$ is its three-dimensional form, defined so that

$$\int f(\mathbf{r})\,\delta(\mathbf{r}-\mathbf{r}_0)\,d\mathbf{r} = f(\mathbf{r}_0),$$

when the integration is taken over a volume in $\mathbf{r}$ co-ordinates which includes the point $\mathbf{r} = \mathbf{r}_0$.

Now consider the expression

$$\tilde{G}\,L[\phi] - \phi\,\tilde{L}[\tilde{G}].$$

By writing out the expression in full and re-grouping the terms it can be shown that

$$\tilde{G}L[\phi] - \phi\tilde{L}[\tilde{G}] = \partial\overline{P}/\partial t + \nabla\,.\,\mathbf{P}, \tag{30}$$

where

$$\overline{P} \equiv \tilde{G}\frac{\partial^2\phi}{\partial t^2} + \phi\frac{\partial^2\tilde{G}}{\partial t^2} - \frac{\partial\phi}{\partial t}\frac{\partial\tilde{G}}{\partial t} + a^2\left\{\tilde{G}\frac{\partial\phi}{\partial t} - \phi\frac{\partial\tilde{G}}{\partial t}\right\} + \nabla\tilde{G}\,.\,\nabla\phi, \tag{31}$$

$$\mathbf{P} \equiv -\tilde{G}\nabla(\partial\phi/\partial t + \phi) - \phi\nabla(\partial\tilde{G}/\partial t - \tilde{G}). \tag{32}$$

Equation (30), which is an appropriate generalization of Green's theorem applicable to the present problem, can now be used to establish a reciprocity relation between the functions G and $\tilde{G}$. To do this, multiply (28) by $\tilde{G}(\mathbf{r}, t \mid \mathbf{r}_1, t_1)$ (note the new source co-ordinates $\mathbf{r}_1$, t_1) and subtract from the resulting quantity the product of $G(\mathbf{r}, t \mid \mathbf{r}_0, t_0)$ with (29) re-written with $\mathbf{r}_1$, t_1 in place of $\mathbf{r}_0$, t_0. Now integrate the result over a fixed volume V, which includes the ends of the source-point vectors $\mathbf{r}_0$ and $\mathbf{r}_1$, and over a time interval from $t = -\infty$ to t', where $t' > t_0$ and t_1. We find that

$$\begin{aligned}
&\int_{-\infty}^{t'} dt \int_V d\mathbf{r}\{\tilde{G}(\mathbf{r}, t \mid \mathbf{r}_1, t_1)\,L[G(\mathbf{r}, t \mid \mathbf{r}_0, t_0)] - G(\mathbf{r}, t \mid \mathbf{r}_0, t_0)\,\tilde{L}[\tilde{G}(\mathbf{r}, t \mid \mathbf{r}_1, t_1)]\} \\
&= -4\pi\int_{-\infty}^{t'} dt \int_V d\mathbf{r}\{\tilde{G}(\mathbf{r}, t \mid \mathbf{r}_1, t_1)\,\delta(\mathbf{r}-\mathbf{r}_0)\,\delta(t-t_0) - G(\mathbf{r}, t \mid \mathbf{r}_0, t_0)\,\delta(\mathbf{r}-\mathbf{r}_1)\,\delta(t-t_1)\} \\
&= -4\pi\{\tilde{G}(\mathbf{r}_0, t_0 \mid \mathbf{r}_1, t_1) - G(\mathbf{r}_1, t_1 \mid \mathbf{r}_0, t_0)\}.
\end{aligned} \tag{33}$$

The last result follows from the property of the δ-functions and the fact that $\mathbf{r}_0$, $\mathbf{r}_1$, and t_0, t_1 all lie within the regions of integration. Now the first integral in (33) contains terms in braces which can be replaced by the expression

$$\partial\overline{P}[\tilde{G}(\mathbf{r}, t \mid \mathbf{r}_1, t_1),\, G(\mathbf{r}, t \mid \mathbf{r}_0, t_0)]/\partial t + \nabla\,.\,\mathbf{P}[\tilde{G}(\mathbf{r}, t \mid \mathbf{r}_1, t_1),\, G(\mathbf{r}, t \mid \mathbf{r}_0, t_0)], \tag{34}$$

from the result written out in (30). We should point out that (30) has been (quite legitimately) modified to the extent that ϕ, which is there a function of $\mathbf{r}$ and t, has been replaced by G written as a function of $\mathbf{r}$ and t and the *fixed* source co-ordinates $\mathbf{r}_0$, t_0. This fact has been indicated in expression (34) by writing in the functional dependence of the scalar $\bar{P}$ and the vector $\mathbf{P}$ on the appropriate G and $\tilde{G}$ functions.

Putting expression (34) into the first integral in (33), we can perform the integration over t on the term $\partial\bar{P}/\partial t$ and, noting that the remaining expression involves the volume integration of the divergence of a vector, use Gauss's theorem to replace this integral by an integration over the surface S (which surrounds V) of the outward normal component of the vector $\mathbf{P}$. The result is

$$\int_V d\mathbf{r}\{\bar{P}[\tilde{G}(\mathbf{r},t' \mid \mathbf{r}_1,t_1), G(\mathbf{r},t' \mid \mathbf{r}_0,t_0)] - \bar{P}[\tilde{G}(\mathbf{r}, -\infty \mid \mathbf{r}_1,t_1), G(\mathbf{r}, -\infty \mid \mathbf{r}_0,t_0)]\} + \int_{-\infty}^{t'} dt \int_S \mathbf{n}.\mathbf{P}[\tilde{G}(\mathbf{r}^s,t \mid \mathbf{r}_1,t_1), G(\mathbf{r}^s,t \mid \mathbf{r}_0,t_0)]\,ds = -4\pi\{\tilde{G}(\mathbf{r}_0,t_0 \mid \mathbf{r}_1,t_1) - G(\mathbf{r}_1,t_1 \mid \mathbf{r}_0,t_0)\}. \quad (35)$$

$\mathbf{n}$ is the unit outwards normal vector to the boundary surface S and $\mathbf{r}^s$ is the value of $\mathbf{r}$ on this same surface. We note that $\mathbf{n}$ is *not* a function of t; S is a *fixed* surface.

Now on physical grounds $G(\mathbf{r},t \mid \mathbf{r}_0,t_0)$ must satisfy a causality condition; i.e., there can be *no* effect felt at a time t earlier than the time t_0 at which the source was initiated. In other words, $G(\mathbf{r},t \mid \mathbf{r}_0,t_0)$ and all of its derivatives must vanish for $t < t_0$. The adjoint function $\tilde{G}(\mathbf{r},t \mid \mathbf{r}_1,t_1)$, since it satisfies a 'time-reversed' equation, must on the other hand vanish identically for all $t > t_1$. Thus the first term in the first integral of (35) vanishes because $t' > t_1$ and $\tilde{G} = 0$, whilst the second term vanishes because $-\infty < t_0$ and $G = 0$.

Taking note of the definition in (32), the integrand of the second integral in (35) can be written as

$$-\tilde{G}\mathbf{n}.\nabla(G + \partial G/\partial t) - G\mathbf{n}.\nabla(\partial\tilde{G}/\partial t - \tilde{G}).$$

$\mathbf{n}.\nabla$ represents a differentiation normal to the boundary surface S and the expression is evaluated for $\mathbf{r} = \mathbf{r}^s$. We note that the operators $\mathbf{n}.\nabla$ and $\partial/\partial t$ are commutative. Thus if $\mathbf{n}.\nabla G = 0$, then $\mathbf{n}.\nabla(\partial G/\partial t) = 0$ too, for example. It can now be seen that if either G and $\tilde{G}$ or $\mathbf{n}.\nabla G$ and $\mathbf{n}.\nabla\tilde{G}$ are zero on the surface S, the integrand is zero and (35) yields the reciprocity relation

$$\tilde{G}(\mathbf{r}_0,t_0 \mid \mathbf{r}_1,t_1) = G(\mathbf{r}_1,t_1 \mid \mathbf{r}_0,t_0). \quad (36)$$

To summarize, (36) will be true if G and $\tilde{G}$ satisfy a causality condition and either homogeneous Dirichlet or homogeneous Neumann conditions on the boundary surface. Just which of these latter conditions should be chosen will be decided by the form of the boundary-value data for the potential ϕ, as we shall see in § 7.

It is now possible to set about the task of finding the general solution for ϕ in terms of the Green function. During the course of this procedure it will become apparent what sort of initial and boundary-value data on ϕ is necessary in order to completely specify any given problem. First of all we define the following functions,

$$\phi_0 \equiv \phi(\mathbf{r}_0,t_0); \quad \tilde{G}_0 \equiv \tilde{G}(\mathbf{r}_0,t_0 \mid \mathbf{r},t), \quad (37a)$$

which satisfy the equations

$$L_0[\phi_0] = -4\pi Q_0; \quad \tilde{L}_0[\tilde{G}_0] = -4\pi\delta(\mathbf{r}-\mathbf{r}_0)\,\delta(t-t_0), \quad (37b)$$

where L_0 and $\tilde{L}_0$ are the operators L and $\tilde{L}$ (defined in (25) and (26)) in terms of $\mathbf{r}_0$, t_0 rather than $\mathbf{r}$, t co-ordinates. Q_0 is the function Q (see (24) and (21)) in $\mathbf{r}_0$, t_0 rather than $\mathbf{r}$, t co-ordinates, and we recall that the δ-function is an even function of its argument.

Using (37*b*) it is easy to see that

$$\int_0^{t+} dt_0 \int_{V_0} d\mathbf{r}_0 \{\tilde{G}_0 L_0[\phi_0] - \phi_0 \tilde{L}_0[\tilde{G}_0]\}$$
$$= -4\pi \int_0^{t+} dt_0 \int_{V_0} d\mathbf{r}_0 (\tilde{G}_0 Q_0) + 4\pi \int_0^{t+} dt_0 \int_{V_0} d\mathbf{r}_0 \{\phi_0 \delta(\mathbf{r}-\mathbf{r}_0)\,\delta(t-t_0)\}, \quad (38)$$

where the integrations (as shown) are taken over the $\mathbf{r}_0$, t_0 space. The upper limit for t_0 is taken as $t+$ to ensure that we integrate 'right across' the $\delta(t-t_0)$ function. Using (30) re-expressed in $\mathbf{r}_0$, t_0 co-ordinates to eliminate the curly bracket terms on the left-hand side of (38) we find that

$$\int_{V_0} d\mathbf{r}_0 \{\bar{P}_0(t_0 = t+) - \bar{P}_0(t_0 = 0)\} + \int_0^{t+} dt_0 \int_{S_0} \mathbf{n}_0 . \mathbf{P}_0^s \, ds_0$$
$$+ 4\pi \int_0^{t+} dt_0 \int_{V_0} d\mathbf{r}_0 (\tilde{G}_0 Q_0) = \begin{cases} 4\pi\phi(\mathbf{r},t), & \text{if } t > 0 \text{ and } \mathbf{r} \text{ is in } V_0, \\ 0, & \text{if } t < 0 \text{ or } \mathbf{r} \text{ is not in } V_0. \end{cases}$$

Here $\bar{P}_0$ and $\mathbf{P}_0$ are $\bar{P}$ and $\mathbf{P}$ (see (31) and (32)) expressed in terms of ϕ_0, $\tilde{G}_0$, $\mathbf{r}_0$ and t_0. S_0 is the surface bounding the volume V_0, and $\mathbf{n}_0$ is the unit outwards vector normal to this (fixed) surface. $\mathbf{P}_0^s$ is the value of $\mathbf{P}_0$ when $\mathbf{r}_0 = \mathbf{r}_0^s$. Since $\bar{P}_0$ contains $\tilde{G}_0 \equiv \tilde{G}(\mathbf{r}_0, t_0 \mid \mathbf{r}, t)$ it follows from the causality condition that $\bar{P}_0(t_0 = t+)$ is zero.

Confining attention to the case $t > 0$ and $\mathbf{r}$ in V_0, it follows that

$$4\pi\phi(\mathbf{r},t) = -\int_{V_0} \bar{P}_0\{\phi(\mathbf{r}_0,0),\, G(\mathbf{r},t \mid \mathbf{r}_0,0)\}\, d\mathbf{r}_0$$
$$+ \int_0^{t+} dt_0 \int_{S_0} \mathbf{n}_0 . \mathbf{P}_0\{\phi(\mathbf{r}_0^s,t_0),\, G(\mathbf{r},t \mid \mathbf{r}_0^s,t_0)\}\, ds_0$$
$$+ 4\pi \int_0^{t+} dt_0 \int_{V_0} G(\mathbf{r},t \mid \mathbf{r}_0,t_0)\, Q_0(\mathbf{r}_0,t_0)\, d\mathbf{r}_0. \quad (39)$$

In deriving (39) we have made use of the particular form of the reciprocity relation (36) which reads

$$\tilde{G}_0 \equiv \tilde{G}(\mathbf{r}_0,t_0 \mid \mathbf{r},t) = G(\mathbf{r},t \mid \mathbf{r}_0,t_0), \quad (40)$$

and we have written in the explicit functional dependences of $\bar{P}_0$ and $\mathbf{P}_0$ on the quantities ϕ and G. The various terms in (39) show that $\phi(r,t)$ depends on the initial conditions (1st integral), the boundary conditions (2nd integral) and the distribution of sources throughout V_0 (3rd integral).

Referring to (32), which defines $\mathbf{P}$, we see that if the boundary conditions on ϕ give $\phi(\mathbf{r}_0^s, t_0)$ as a known function of $\mathbf{r}_0^s$ and t_0, we should choose $G(\mathbf{r},t \mid \mathbf{r}_0^s,t_0) = 0$ on S_0 whilst if the boundary value data on ϕ is of the form $\mathbf{n}_0 . \nabla_0 \phi(\mathbf{r}_0^s, t_0)$ as a given function of $\mathbf{r}_0^s$ and t_0, we should choose $\mathbf{n}_0 . \nabla_0 G(\mathbf{r},t \mid \mathbf{r}_0^s,t_0) = 0$. Together with the causality condition, this information is sufficient to determine G and hence, through (39), the function ϕ. We note that the data on G is given in $\mathbf{r}_0$, t_0 co-ordinates, so that we should use the second of equations (37*b*) together with the reciprocity relation written out in (40) in order to find G. That is to say, we should solve the equation

$$\tilde{L}_0[G(\mathbf{r},t\,|\,\mathbf{r}_0,t_0)] = -4\pi\delta(\mathbf{r}-\mathbf{r}_0)\,\delta(t-t_0), \tag{41}$$

with the conditions just discussed, to find G.

4. Unbounded domain Green's function

In this section we shall derive the Green function for an unbounded, three-dimensional domain. To distinguish it from the more general function G, used above, we shall write it as

$$g = g(\mathbf{r},t\,|\,\mathbf{r}_0,t_0), \tag{42}$$

satisfying the equation (see (41) and (26)),

$$-\frac{\partial^3 g}{\partial t_0^3}+\nabla_0^2\frac{\partial g}{\partial t_0}+a^2\frac{\partial^2 g}{\partial t_0^2}-\nabla_0^2 g = -4\pi\delta(\mathbf{r}-\mathbf{r}_0)\,\delta(t-t_0). \tag{43}$$

We now define $\bar{g}$, the Fourier transform of g, so that

$$\bar{g}(\mathbf{r},t\,|\,\mathbf{r}_0;\,\zeta) = \frac{1}{\surd(2\pi)}\int_{-\infty}^{\infty} g(\mathbf{r},t\,|\,\mathbf{r}_0,t_0)\,e^{i\zeta t_0}\,dt_0. \tag{44}$$

Taking note of the fact that g must be zero for $t_0 > t > -\infty$, we can assume that $|g| \leqslant A\,e^{-\epsilon' t_0}$ for some $\epsilon' > 0$ as $t_0 \to -\infty$, where A is some positive constant. Convergence of the integral for $\bar{g}$ as $t_0 \to -\infty$ is then secured if ζ has an imaginary part $-i\epsilon$ such that $\epsilon > \epsilon' > 0$. Then g can be found from the inversion formula

$$g(\mathbf{r},t\,|\,\mathbf{r}_0,t_0) = \frac{1}{\surd(2\pi)}\int_{-\infty-i\epsilon}^{\infty-i\epsilon} \bar{g}(\mathbf{r},t\,|\,\mathbf{r}_0;\,\xi)\,e^{-it_0\xi}\,d\xi. \tag{45}$$

Multiplying (43) by $e^{i\zeta t_0}/\surd(2\pi)$ and integrating with respect to t_0 from $-\infty$ to $+\infty$, it can be shown that $\bar{g}$ satisfies the following equation,

$$\nabla_0^2\bar{g}+\chi^2\bar{g} = 4\pi f\delta(\mathbf{r}-\mathbf{r}_0), \tag{46}$$

where

$$\chi^2 \equiv \zeta^2\left(\frac{\zeta-ia^2}{\zeta-i}\right), \tag{47}$$

$$f \equiv \frac{-i\,e^{i\zeta t}}{(\zeta-i)\surd(2\pi)}. \tag{48}$$

Before proceeding to solve (46), let us investigate the behaviour of $\bar{g}$ as $|\mathbf{r}_0-\mathbf{r}| \to 0$. Integrating over a small spherical volume surrounding the field point $\mathbf{r}$ and using Gauss's theorem to deal with the first term, we find that

$$\int \boldsymbol{\nabla}_0\,\bar{g}\,.\,d\mathbf{s}_0+\chi^2\int \bar{g}\,d\mathbf{r} = 4\pi f, \tag{49}$$

where $\boldsymbol{\nabla}_0\,\bar{g}$ is the vector gradient of $\bar{g}$ and $d\mathbf{s}_0$ represents the (vector) element of surface area. The result on the right-hand side arises from the properties of the function $\delta(\mathbf{r}-\mathbf{r}_0)$. The function $\bar{g}$ will be independent of the spherical polar angles, i.e. it will depend only on $R = |\mathbf{r}_0-\mathbf{r}|$, so that $\boldsymbol{\nabla}_0\,\bar{g}$ will be in the radial direction, parallel to $d\mathbf{s}_0$, and will have the same magnitude everywhere over the surface. Thus the first integral in (49) will be equal to $d\bar{g}/dR$ times the area of the sphere, namely $4\pi R^2$. It is apparent that as $R \to 0$ the second term in (49) becomes negligible and we find that

$$\left(\frac{d\bar{g}}{dR}\right)_{R\to 0} \to \frac{f}{R^2}$$

or, finally,

$$(\bar{g})_{R\to 0} \to -\frac{f}{R}. \tag{50}$$

In setting out to solve (46) it will be convenient to move the origin of co-ordinates to the field point $\mathbf{r}$ and to define the spherical polar co-ordinate system $R = |\mathbf{r}_0 - \mathbf{r}|$, ω_0 and ϑ_0, where $0 \leqslant \omega_0 \leqslant 2\pi$ and $-\frac{1}{2}\pi \leqslant \vartheta_0 \leqslant \frac{1}{2}\pi$. A volume element in this system is given by $R^2 \cos\vartheta_0\, dR\, d\vartheta_0\, d\omega_0$, and we infer that $\delta(\mathbf{r}-\mathbf{r}_0)$ must take the form

$$\frac{2\delta(R)\,\delta(\vartheta-\vartheta_0)\,\delta(\omega-\omega_0)}{R^2\cos\vartheta_0}$$

in this system. Multiplying (46) by $\cos\vartheta_0\, d\vartheta_0\, d\omega_0$ and integrating over 0 to 2π in ω_0 and $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$ in ϑ_0 leads to the result that

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{d\bar{g}}{dR}\right) + \chi^2\bar{g} = \frac{2\delta(R)}{R^2}f. \tag{51}$$

We have again made use of the fact that $\bar{g}$ is not a function of ω_0 or ϑ_0 in the unbounded domain, and have written out the remaining part of ∇_0^2 in full. It is more convenient to solve for $R\bar{g}$ rather than $\bar{g}$ alone, and it is easy to show that

$$\frac{d^2}{dR^2}(R\bar{g}) + \chi^2(R\bar{g}) = \frac{2\delta(R)}{R}f. \tag{52}$$

The general solution of (52) can be written down as follows

$$R\bar{g} = A\,e^{-i\chi R} + B\,e^{i\chi R} - f\frac{e^{-i\chi R}}{i\chi}\int^{R}\frac{\delta(R)}{R}e^{i\chi R}\,dR + f\frac{e^{i\chi R}}{i\chi}\int^{R}\frac{\delta(R)}{R}e^{-i\chi R}\,dR, \tag{53}$$

since the Wronskian of the homogeneous solutions is $-i2\chi$. It remains for us to select the two 'constants' A and B and the appropriate limits for the two integrals involved in such a way as to give a solution of $\bar{g}$ which not only behaves like (50), but also satisfies the causality condition. The latter can be effected by dropping the terms in $e^{i\chi R}$, i.e. by setting $B = 0$ and choosing, say, $+\infty$ for the lower limit in the last integral of (53). This integral then vanishes for all $R > 0$ because $\delta(R)$ is zero in these circumstances. In the first integral of (53) we shall choose the lower limit to be $-\infty$, so that we have to deal with the solution

$$R\bar{g} = Ae^{-i\chi R} - f\frac{e^{-i\chi R}}{i\chi}\int_{-\infty}^{R}\frac{\delta(\alpha)}{\alpha}e^{i\chi\alpha}\,d\alpha.$$

Expanding $e^{i\chi\alpha}$, we readily see that

$$R\bar{g} = Ae^{-i\chi R} - f\frac{e^{-i\chi R}}{i\chi}\int_{-\infty}^{R}\left\{1 + i\chi\alpha + \dots\frac{(i\chi)^n\alpha^n}{n!} + \dots\right\}\frac{\delta(\alpha)}{\alpha}\,d\alpha, \tag{54}$$

$$= Ae^{-i\chi R} - f\,e^{-i\chi R}. \tag{55}$$

The first term of the integral in (54) vanishes because it is an integral of an odd function, and all the terms in α^n vanish because $n \geqslant 2$. The result is as written in (55) (remember $R > 0$); comparing it with condition (50) we see that A must be zero and the desired solution for $\bar{g}$ is simply

$$\bar{g} = -\frac{f}{R} e^{-i\chi R} \tag{56}$$

for all $R > 0$.

Using (45) and writing out f and χ in full (see equations (47) and (48)) gives

$$g(\mathbf{r}, t \mid \mathbf{r}_0, t_0) = \frac{i}{2\pi R} \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \exp\left\{ i\xi(t-t_0) - i\xi \sqrt{\frac{\xi - ia^2}{\xi - i}}\, R \right\} \frac{d\xi}{\xi - i}. \tag{57}$$

We remark that throughout χ is taken to be that root of χ^2 which behaves like $+\sqrt{|\xi|}$ as $\xi \to +\infty$.

5. The behaviour of g

As a check on the result given for g above, consider the special case $a = 1$. Equation (43) can then be written as

$$\left(\frac{\partial^2}{\partial t_0^2} - \nabla_0^2\right)\left(-\frac{\partial g}{\partial t_0} + g\right) = -4\pi\delta(\mathbf{r} - \mathbf{r}_0)\,\delta(t - t_0).$$

It follows that $-\partial g/\partial t_0 + g$ should be the same as the Green function for the familiar scalar wave equation. We see that

$$\begin{aligned} -\frac{\partial g}{\partial t_0} + g &= -\frac{1}{2\pi R} \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \exp\{-i\xi[R - (t - t_0)]\}\, d\xi \\ &= -\frac{1}{R}\,\delta(R - (t - t_0)), \end{aligned} \tag{58}$$

which is indeed the result that we should expect.

Returning to (57), we observe that the contour $\pm\infty - i\epsilon$ can be closed in the lower half of the ξ-plane by a semi-circle of infinite radius without surrounding any singularities of the integrand. Accordingly, we can re-write

$$g(\mathbf{r}, t \mid \mathbf{r}_0, t_0) = \frac{i}{2\pi R} \int_{-\pi+}^{0-} \exp\{[i\,|\xi| \cos\theta - |\xi| \sin\theta]\,[(t - t_0) - R] - \tfrac{1}{2}(a^2 - 1)\, R\} \times \left\{1 + O\left(\frac{1}{|\xi|}\right)\right\} i\, d\theta$$

as $|\xi| \to \infty$ (N.B. $\xi = |\xi|\, e^{i\theta}$). It follows that g is zero for all $t_0 > t - R$. Thus g satisfies the causality condition because $R > 0$. In addition, we note that the wave-front is of spherical form about the source point $\mathbf{r}_0$ and travels with 'unit speed' (i.e. at the frozen sound speed $a_{f\infty}$, in dimensional terms).

Expanding parts of the integrand of (57) in inverse powers of ξ enables us to write

$$g = \frac{i}{2\pi R} \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \exp\{-i\xi[R - (t - t_0)]\} \exp\{-\tfrac{1}{2}(a^2 - 1) R\} \left(1 + O\left(\frac{R}{\xi}\right)\right)\left(1 - \frac{i}{\xi}\right)^{-1} \frac{d\xi}{\xi},$$

so that when $R \to (t - t_0)$ from below

$$g \to -\frac{\exp\{-\tfrac{1}{2}(a^2 - 1) R\}}{R}. \tag{59}$$

The spherical wave-front expanding with unit velocity around the source point is therefore in the nature of a step function. The amplitude of the step decreases as a result of both spherical attenuation (the factor R^{-1}) and relaxation attenuation, or absorption, as summarized in the term $\exp[-\frac{1}{2}(a^2-1)R]$.

If we re-arrange the integral in (57) so that

$$g = \frac{i}{2\pi R}\int_{-\infty-i\epsilon}^{\infty-i\epsilon} \exp\left\{-i\xi[aR-(t-t_0)]-i\xi\left\{\sqrt{\frac{\xi-ia^2}{\xi-i}}-a\right\}R\right\}\frac{d\xi}{\xi-i}.$$

it can be shown that the function multiplied by R in the exponential has a saddle point at $\xi = 0$ which yields the dominant contribution to g. The method of steepest descents then gives

$$g \sim -\frac{1}{R^{\frac{3}{2}}}\frac{\exp\{-[aR-(t-t_0)]^2/2R(a-a^{-1})\}}{\sqrt{\{2\pi(a-a^{-1})\}}}, \tag{60}$$

as $R \to \infty$ in the vicinity of $aR-(t-t_0) = 0$. The latter relation represents the location of a spherical front which expands from the source point with the equilibrium sound speed a^{-1}, in terms of the unit speed, or $a_{e\infty}$ in terms of dimensional speeds.

The two results exhibited in (59) and (60) show that, after a sufficient lapse of time, the potential function arising from the source differs from zero throughout a sphere of radius $R = t-t_0$, with its peak value occurring where $R = (t-t_0)/a$. It is interesting to note that the amplitude of this peak decays like $R^{-\frac{3}{2}}$; in other words, the usual spherical attenuation is reinforced by the absorptive relaxation processes which are of a 'square root' character in this region.

The case $a = 1$ considered above has given us (in (58)) the Green function for a non-dispersive, non-absorbing medium and we note the contrast between its behaviour and the behaviour of g in the relaxing gas case. In the first place the relaxation processes spread the disturbances from the source throughout the whole sphere $R = (t-t_0)$, instead of them being confined within the infinitesimally thin shell, as in the case $a = 1$. In the second place, the quantity R times the Green function is finite for all $t > t_0$ in the relaxing gas case, although a hint of the original δ-function input remains at the location $R = (t-t_0)/a$, as is shown by the result in (60) for large times after initiation of the source. We note that (57) gives

$$(Rg)_{R\to 0} \to -e^{-(t-t_0)} \quad (t>t_0); \tag{61}$$

so that the potential in the vicinity of the source subsides exponentially with increasing time back to a final value of zero as $(t-t_0) \to \infty$.

Some physical interpretation of g can be derived from (39). If there is no boundary surface S_0 and if the system is initially quiescent (so that $\phi(\mathbf{r},0)$ and all of its derivatives are zero) the first two integrals vanish and we are left with

$$\phi(\mathbf{r},t) = \int_0^{t+} dt_0 \int_{V_0} g(\mathbf{r},t\,|\,\mathbf{r}_0,t_0)\,Q_0(\mathbf{r}_0,t_0)\,d\mathbf{r}_0. \tag{62}$$

If $Q_0(\mathbf{r}_0,t_0) = \delta(\mathbf{r}_0)\,\delta(t_0+)$ it follows that

$$\phi(\mathbf{r},t) = g(\mathbf{r},t\,|\,0,0+) \tag{63}$$

and g can be identified as the potential produced by a δ-pulse of heat. Referring to the definition of the function Q ((21) and (24)), it can be seen that, if $q = q_2$, then

$$Q_0 = \tau(\gamma_e - 1)\, q_2(\mathbf{r}_0, t_0)/4\pi a^2_{e\infty},$$

and the δ-pulse of heat is a pulse *added to the internal mode only*. Consequently it is this form of heat addition which gives rise to the fundamental type of source flow.

6. Motion in the unbounded domain

When there are no solid surfaces present in the field of flow the general solution for the potential is given by the first and third integrals of (39) only, with the general function G replaced by the function g discussed in the two previous sections. In order to illustrate solutions of this type we shall deal briefly with two special cases. First, we shall examine the source-free field ($Q_0 = 0$) which results from prescribed initial conditions and, secondly, we shall look at the field created by a distribution of sources initiated at the time $t = 0+$ in a quiescent gas.

The first problem is of the initial-value type, and its solution is

$$4\pi\phi(\mathbf{r}, t) = -\int \overline{P}_0\{\phi(\mathbf{r}_0, 0), g(\mathbf{r}, t \mid \mathbf{r}_0, 0)\}\, d\mathbf{r}_0, \tag{64}$$

where the integration extends over all space. Writing out the scalar function $\overline{P}_0$ in full (see (31)) gives

$$4\pi\phi(\mathbf{r}, t) = -\int\left\{g\frac{\partial^2\phi_0}{\partial t_0^2} + \phi_0\frac{\partial^2 g}{\partial t_0^2} - \frac{\partial\phi_0}{\partial t_0}\frac{\partial g}{\partial t_0} + a^2\left[g\frac{\partial\phi_0}{\partial t_0} - \phi_0\frac{\partial g}{\partial t_0}\right] + \nabla_0 g\,.\,\nabla_0\,\phi_0\right\} d\mathbf{r}_0, \tag{65}$$

where all the quantities inside the braces are evaluated at time $t_0 = 0$. It is apparently necessary to specify all of ϕ_0, $\partial\phi_0/\partial t_0$, $\partial^2\phi_0/\partial t_0^2$ and $\nabla_0\,\phi_0$ at this time and for all $\mathbf{r}_0$ in order to solve the problem.

From the definition of ϕ it is clear that specification of $\partial\phi/\partial t$ and $\nabla\phi$ is equivalent to prescribing the values of the pressure and velocity perturbations, respectively. In order to identify $\partial^2\phi/\partial t^2$ we can return to (12). Putting q and q_2 equal to zero in (12) and (9), linearizing and non-dimensionalizing as in §2, it readily follows that

$$\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = C_2(\gamma_1 - 1)\frac{C_{p1}}{C_p}\frac{(T_1 - T_2)}{a^2_{f\infty}}. \tag{66}$$

Therefore it can be seen that the value of $\partial^2\phi/\partial t^2$ is intimately connected with the difference between T_1 and T_2. In other words, with the extent of the lack of equilibrium at any particular place and time.

Naturally one cannot select the initial values required in (65) with complete arbitrariness, for choosing $\phi(\mathbf{r}_0, 0)$ will fix $\nabla_0\,\phi(\mathbf{r}_0, 0)$ and hence $\nabla_0^2\phi(\mathbf{r}, 0)$. Then, through (66), the value of $\partial^2\phi/\partial t_0^2$ will be influenced by the choice of initial values for T_1 and T_2. The selection of a value for $\partial\phi_0/\partial t_0$ remains open, however.

An interesting case occurs if we choose $\phi(\mathbf{r}_0, 0) = 0 = \partial\phi(\mathbf{r}_0, 0)/\partial t_0$. It follows from the first of these that $\nabla_0\,\phi(\mathbf{r}_0, 0) = 0 = \nabla_0^2\,\phi(\mathbf{r}_0, 0)$, and thence from (66) that the choice of $\partial^2\phi(\mathbf{r}_0, 0)/\partial t_0^2$ is equivalent to the choice of a particular degree of out-of-equilibrium at the initial instant. The initial field is one of zero velocity and pressure perturbations and, if T_1 should equal T_2 as well, it is clear that $\phi(\mathbf{r},t) = 0$; i.e. the initial field is in equilibrium and remains so for all time. In general, however,

$$4\pi\phi(\mathbf{r},t) = -\int g \frac{\partial^2\phi_0}{\partial t_0^2} d\mathbf{r}_0, \tag{67}$$

and if we select

$$\frac{\partial^2\phi_0}{\partial t_0^2} = -4\pi\delta(\mathbf{r}_0),$$

it follows that

$$\phi(\mathbf{r},t) = g(\mathbf{r},t \mid 0,0). \tag{68}$$

The similarity between this result and that in (63) above is apparent, and we identify the addition of a δ-pulse of heat into the internal mode alone with a value of T_2 instantaneously an infinite amount greater than the initial, ambient, value of T_1 at some isolated point in the field (see (66)).

We have already written down the solution for the potential of a field created by a distribution of sources from a quiescent state, existing prior to $t = 0$, in (62) in the previous section. To reiterate

$$\phi(\mathbf{r},t) = \int_0^{t+} dt_0 \int g(\mathbf{r},t \mid \mathbf{r}_0,t_0)\, Q_0(\mathbf{r}_0,t_0)\, d\mathbf{r}_0.$$

When $q = q_2$ the potential depends on q only, as we have seen, but if $q \neq q_2$ and, in consequence, some heat is added directly to the translational states, then ϕ depends also on how rapidly the rate of heat addition to these states varies with time, as witnessed by the appearance of the term $\partial(q-q_2)/\partial t$ in Q.

Whilst it is comparatively reasonable to visualize the direct gain or loss of energy from the internal mode as a result of some photo-process, the corresponding physical processes which lead to direct changes of translational state are somewhat more difficult to describe in the context of the present theory. Usually $q-q_2$ is ascribed to an idealization of the local heat release by chemical reaction, but to do so in the present case would hardly be warranted, since a reaction is itself another relaxation-type of process and should, strictly, be included as such in our theory. We shall not pursue the case $q \neq q_2$ any further for these reasons, merely remarking in conclusion that if it should be found that the idealization referred to is physically acceptable, (62) provides the desired, infinite domain, solution.

7. The spherical piston problem

We shall end with a discussion of a simple problem involving the presence of a solid surface in the flow field. With conditions of initial quiescence and no heat sources in the field, (39) shows that

$$4\pi\phi(\mathbf{r},t) = -\int_0^{t+} dt_0 \int_{S_0} G(\mathbf{r},t \mid \mathbf{r}_0^s,t_0)\, \mathbf{n}_0 . \nabla_0 \left\{\phi(\mathbf{r}_0^s,t_0) + \frac{\partial}{\partial t_0}\phi(\mathbf{r}_0^s,t_0)\right\} ds_0, \tag{69}$$

provided that

$$\mathbf{n}_0 . \nabla_0 G(\mathbf{r},t \mid \mathbf{r}_0^s,t_0) = 0. \tag{70}$$

If the boundary surface S_0 is a sphere with centre at $\mathbf{r}_0 = 0$, i.e.

$$r_0^s = R^s = \text{const.}, \tag{71}$$

then $\mathbf{n}_0 . \boldsymbol{\nabla}_0$ becomes simply $-\partial/\partial r_0$ if we consider the field *external* to S_0. In the case of the problem whose solution is given in (69) and (70), we need to prescribe $\partial\phi_0/\partial r_0$ on $r_0 = R^s$, so that in physical terms we are dealing with the case of a spherically expanding or contracting piston. Let us write

$$\frac{\partial\phi_0}{\partial r_0} = U(t_0)\,H(t_0) \quad \text{when} \quad r_0 = r_0^s = R^s, \tag{72}$$

where $U(t_0)$ is a finite continuous function of t_0 and $H(t_0)$ is the unit step function. It follows that

$$\frac{\partial^2\phi_0}{\partial r_0\,\partial t_0} = U'(t_0)\,H(t_0) + U(0)\,\delta(t_0) \quad (r_0 = R^s). \tag{73}$$

From physical considerations the solution must be independent of the spherical polar angles, i.e. it must depend on r only. We can therefore look for a value of G which satisfies (41) (in which only the r_0-derivatives in the operator $\tilde{L}_0$ survive) along with the condition 70 and the condition of causality. If we write $\bar{G}$ for the Fourier transform of G in the same way that $\bar{g}$ stands for the transform of g (see (44) and (45)), this means that $\bar{G}$ satisfies the equation

$$\frac{1}{r_0^2}\frac{d}{dr_0}\left(r_0^2\frac{d\bar{G}}{dr_0}\right) + \chi^2\bar{G} = \frac{\delta(r-r_0)}{r_0^2}f. \tag{74}$$

(χ^2 and f are defined in (47) and (48) respectively.)

The general solution of (74) is quite similar to (53) for $\bar{g}$, and if we select the limits for the integrals so that the causality condition is satisfied, we can write

$$r_0\bar{G} = A\,e^{-i\chi r_0} - f\frac{e^{-i\chi r_0}}{i2\chi}\int_0^{r_0}\frac{\delta(r-r')}{r'}e^{i\chi r'}\,dr' - f\frac{e^{i\chi r_0}}{i2\chi}\int_{t_0}^{\infty}\frac{\delta(r-r')}{r'}e^{-i\chi r'}\,dr'. \tag{75}$$

The 'constant' A must now be found so that condition (70) is satisfied; in particular, $\partial\bar{G}/\partial r_0 = 0$ when $r_0 = R^s$ and $r > r_0$, since we are dealing with the region exterior to the sphere $r_0 = R^s$. It readily transpires that

$$\bar{G}(r,t\,|\,r_0;\zeta) = \frac{f\exp\{-i\chi(r-r_0)\}}{i2\chi r r_0}\left\{\exp\{i2\chi(R^s-r_0)\}\left[\frac{1-i\chi R^s}{1+i\chi R^s}\right]-1\right\}. \tag{76}$$

The particular value of $\bar{G}$ required in the solution, (69), is found by letting $r_0 \to R^s$, and is

$$\bar{G}(r,t\,|\,R^s;\zeta) = \frac{-f\exp\{-i\chi(r-R^s)\}}{r(1+i\chi R^s)} \quad (r > R^s). \tag{77}$$

Inverting the transform gives

$$G(r,t\,|\,R^s,t_0) = -\frac{1}{2\pi r}\int_{-\infty-i\epsilon}^{\infty-i\epsilon}\frac{\exp\{i\xi(t-t_0)-i\chi(r-R^s)\}}{(\xi-i)\,(1+i\chi R^s)}\,d\xi, \tag{78}$$

from which it follows that $G = 0$ for $t_0 \geqslant t-(r-R^s)$.

Since G and ϕ are not functions of the spherical polar angles, the surface integration in (69) can be carried out and the solution can be written down as follows

$$\phi(r,t) = R^{s^2}\left\{U(0)\,G(r,t\,|\,R^s,0) + \int_{0+}^{t-(r-R^s)}[U(t_0)+U'(t_0)]\,G(r,t\,|\,R^s,t_0)\,dt_0\right\}. \tag{79}$$

An especially simple case occurs when $U'(t_0) = 0$; then

$$U(t_0) = U(0) = U = \text{const.}$$

and we find that

$$\phi(r,t) = -\frac{UR^{s^2}}{i2\pi r}\int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{\exp\{i\xi t - i\chi(r-R^s)\}}{(1+i\chi R^s)}\frac{d\xi}{\xi}. \tag{80}$$

This then is the potential for the flow created by a spherical piston expanding at a constant surface speed (contracting piston if $U < 0$) having started impulsively from rest at the initial instant. As such it is the three-dimensional analogue of the one-dimensional piston problem studied by Chu (1957).

Some interest attaches to the jump in density which occurs across the wave head, located, as can be seen from (80), at $r = R^s + t$. The linearized and non-dimensionalized version of (1) shows that

$$\frac{\rho-\rho_\infty}{\rho_\infty} = -\int^t \nabla^2\phi\, dt + F(r),$$

in the spherically symmetric case. Since $\rho = \rho_\infty$ prior to $t = 0$ the arbitrary function $F(r)$ must be zero. It can then be shown that

$$\frac{\rho-\rho_\infty}{\rho_\infty} = \frac{UR^{s^2}}{2\pi r}\int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{\exp\{i\xi t - i\chi(r-R^s)\}}{(1+i\chi R^s)}\left(\frac{\xi - ia^2}{\xi - i}\right) d\xi. \tag{81}$$

Expanding the integrand for large $|\xi|$ and letting $t-(r-R^s) \to 0$ from above shows that

$$\frac{\rho-\rho_\infty}{\rho_\infty} = \frac{UR^s}{r}\exp\{-\tfrac{1}{2}(a^2-1)(r-R^s)\}; \quad t-(r-R^s) = 0+. \tag{82}$$

The jump in density on crossing the wave head therefore decays exponentially with the time as a result of the relaxation effects and also like $1/r$ as a result of spherical attenuation. We observe that in the early stages of the piston's expansion, when $(R^s/r) \simeq 1$, the density behaves in the same way as it does in the one-dimensional case mentioned above, as indeed we should expect it to do.

REFERENCES

Chu, B.-T. 1957 *Brown University, Wright Air Development Centre*, TN 57–213.

Clarke, J. F. & McChesney, M. 1964 *The Dynamics of Real Gases*. London: Butterworths.

Vincenti, W. G. 1959 *J. Fluid Mech.* **6**, 481–96.